Multidimensional generalized superalgebras

1992 J. Phys. A: Math. Gen. 25 L511

(http://iopscience.iop.org/0305-4470/25/9/001)

J. Phys. A: Math. Gen. 25 (1992) L511–L514. Printed in the UK

LETTER TO THE EDITOR

Multidimensional generalized superalgebras

V V Semenov

Institute of Spectroscopy, USSR Academy of Science, Troitsk, Moscow Region 142092, Russia

Received 18 November 1991, in final form 18 February 1992

Abstract. The multidimensional analogues of the one-dimensional generalized superalgebras are considered in the framework of quantum mechanics.

In [1] the new approach of constructing the various one-dimensional generalizations of the one-dimensional supersymmetric quantum mechanics was proposed. The basis of our approach is an investigation of a simple model which describes a non-interacting N-level system and one bosonic mode. The case of the two-level system leads to ordinary supersymmetric quantum mechanics [2]. In the case of multilevel systems $(N \geqslant 3)$ there exist many possible schemes of configurations of the energy levels. So, the schemes of non-degenerate levels correspond to parasupersymmetric quantum mechanics [3, 4]. But for schemes of degenerate levels one can construct the generalized superalgebras differing from the parasuperalgebras.

In this letter we extend our method in order to obtain multidimensional generalized superalgebras. Note that, for ordinary supersymmetric quantum mechanics, there exist two approaches of multidimensional extensions (so-called standard [5] and the spin–orbit coupling [6–8] procedures). The general feature of the standard and the spin–orbit coupling procedures is the increase of number of bosonic degrees of freedom, or bosonic modes, simultaneously with the increase (although different in each case) of fermionic degrees of freedom. In general, these approaches can be utilized for the generalized superalgebras including parasuperalgebras. However, as we shall show further for the generalized superalgebras based on multilevel systems, another approach exists in which multidimensional extensions are realized by an increase of the number of bosonic modes only (number of 'discrete' degrees of freedom does not change). For this purpose, let us consider a simple system consisted of the noninteracting N-level system and K bosonic modes. Certainly, we suppose that the frequencies ω_k of bosonic modes are equal to the distances between levels of the N-level system. The Hamiltonian of such system has the following form:

$$H=\frac{1}{2}\sum_{k=1}^{K}\omega_k\{b_k, b_k^+\}+h \tag{1}$$

where operators b_k^+ and b_k are the ordinary bosonic creation and annihilation operators obeying commutation rules: $[b_i, b_j]=[b_i^+, b_j^+]=0$; $[b_i, b_j^+]=\delta_{ij}$. Its infinite-dimensional representations are

$$b_k=(2\omega_k)^{-1/2}(p_k-\mathrm{i}\omega_k x_k) \qquad b_k^+=(2\omega_k)^{-1/2}(p_k+\mathrm{i}\omega_k x_k). \tag{2}$$

As for the matrix h, it has the following form for an N-level system: $h_{\alpha\beta}=\varepsilon_\alpha\delta_{\alpha\beta}$, $a, \beta=1,\ldots,N$, and we suppose that the relations $\varepsilon_\gamma=\varepsilon_\delta=\omega_i$ are satisfied for some γ, δ and i.

The possibility of multidimensional extension will now be exemplified by the non-interacting three-level system and two bosonic modes of different frequencies. In this case the three configurations energy levels are possible (see figure 1) [9]. Accordingly, the matrix elements $h_{\alpha\beta}$ and frequencies ω_k of bosonic modes satisfy the conditions: $\varepsilon_1 = \varepsilon_2 = \omega_1$, $\varepsilon_2 - \varepsilon_3 = \omega_2$ for Ξ-type, $\varepsilon_1 = \varepsilon_2 = \omega_1$, $\varepsilon_3 - \varepsilon_2 = \omega_2$ for V-type and $\varepsilon_2 - \varepsilon_1 = \omega_1$, $\varepsilon_2 - \varepsilon_3 = \omega_2$ for Λ-type.

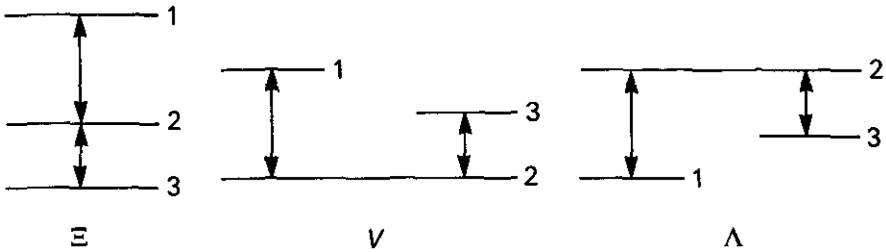

Figure 1. Possible schemes of configurations of the energy levels in a three-level system (case of two bosonic modes of different frequencies).

Firstly, consider a three-level system of Ξ-type and two bosonic modes. Note that this case is a two-dimensional extension of second-order parasupersymmetric quantum mechanics [3, 4]. The Hamiltonian has the following form:

$$H = \frac{1}{2} \sum_{k=1}^{2} \omega_k \{b_k, b_k^+\} + h \tag{3}$$

where $h = \mathrm{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $\varepsilon_1 - \varepsilon_2 = \omega_1$, $\varepsilon_2 - \varepsilon_3 = \omega_2$.

Since the states $|\varepsilon_1\rangle|n_1 - 1, n_2 - 1\rangle$, $|\varepsilon_2\rangle|n_1, n_2 - 1\rangle$, $|\varepsilon_3\rangle|n_1, n_2\rangle$ have equal energy, the operators (generalized supercharge) which are the integrals of motion and have the structure $Q^- \propto ab^+$ and $Q^+ \propto a^+ b$, transform the system from one state to another and back. The operators $a_1^\pm$ are the transition operators between levels 1 and 2 and the operators $a_2^\pm$ are the transition operators between levels 2 and 3 of the Ξ-type system. For operators a_i^+ and a_i^- we find:

$$a_1^+ = \left(\frac{1}{2}\right)^{1/2} \begin{vmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad a_2^+ = \left(\frac{1}{2}\right)^{1/2} \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{vmatrix}. \tag{4}$$

where x and y are arbitrary real non-zero numbers.

We define the generalized supercharges as:

$$Q_1^+ = (\omega_1)^{1/2} \begin{vmatrix} 0 & b_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad Q_2^+ = (\omega_2)^{1/2} \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{vmatrix}. \tag{5}$$

After changing the zero point energy scale, the Hamiltonian (3) is presented in the following form:

$$H = \frac{1}{2} \sum_{k=1}^{2} \omega_k \{b_k, b_k^+\} + \tfrac{1}{2} V \tag{6}$$

where $V = \mathrm{diag}(\omega_1 + \omega_2, -\omega_1 + \omega_2, -\omega_1 - \omega_2)$.

The Hamiltonian (6) and the generalized supercharges (5) satisfy the relations:

$$
\begin{aligned}
&Q_i^{-2} = Q_i^{+2} = [H, Q_i^-] = [H, Q_i^+] = 0 \qquad (i, j = 1, 2)\\
&Q_1^+(Q_1^-Q_1^+ + Q_2^+Q_2^-) = Q_1^+H \\
&(Q_1^-Q_1^+ + Q_2^+Q_2^-)Q_2^+ = Q_2^+H.
\end{aligned}
\tag{7}
$$

(plus Hermitian conjugated relations). Other triple products of the generalized supercharges (5) vanish.

Superpotentials can be introduced as usual

$$
Q_1^+ = \left(\frac{1}{2}\right)^{1/2} \begin{vmatrix} 0 & p_x - \mathrm{i}W_1(x) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad Q_2^+ = \left(\frac{1}{2}\right)^{1/2} \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & p_y - \mathrm{i}W_2(y) \\ 0 & 0 & 0 \end{vmatrix} \tag{8}
$$

where $W_1(x)$ and $W_2(y)$ are arbitrary functions.

The Hamiltonian is defined as:

$$
H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}(W_1^2(x) + W_2^2(y)) + \frac{1}{2}\begin{vmatrix} W_2'(x) + W_2'(y) & 0 & 0 \\ 0 & -W_1'(x) + W_2'(y) & 0 \\ 0 & 0 & -W_1'(x) - W_2'(y) \end{vmatrix} \tag{9}
$$

where $w_1'(x) = \mathrm{d}W_1(x)/\mathrm{d}x$ and $W_2'(y) = \mathrm{d}W_2(y)/\mathrm{d}y$. The Hamiltonian (9) and generalized supercharges (8) satisfy the algebra (7). This algebra is the two-dimensional analogue of the one-dimensional second-order parasuperalgebra [3, 4].

Analogously, one can consider V-type and Λ-type systems. For these cases we give results in a final form. For V-type we obtain that the generalized supercharges have the form:

$$
Q_1^+ = \left(\frac{1}{2}\right)^{1/2} \begin{vmatrix} 0 & p_x - \mathrm{i}W_1(x) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad Q_2^+ = \left(\frac{1}{2}\right)^{1/2} \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p_y - \mathrm{i}W_2(y) & 0 \end{vmatrix}. \tag{10}
$$

The Hamiltonian is

$$
H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}(W_1^2(W_1^2(x) + W_2^2(y)) + \frac{1}{2}\begin{vmatrix} W_1'(x) - W_2'(y) & 0 & 0 \\ 0 & -W_1'(x) - W_2'(y) & 0 \\ 0 & 0 & -W_1'(x) + W_2'(y) \end{vmatrix} \tag{11}
$$

where $W_1'(x) = \mathrm{d}W_1(x)/\mathrm{d}x$ and $W_2'(y) = \mathrm{d}W_2(y)/\mathrm{d}y$. The Hamiltonian (11) and the generalized supercharges (10) generate the algebra:

$$
\begin{aligned}
&Q_i^{-2} = Q_i^{+2} = [H, Q_i^-] = [H, Q_i^+] = 0 \qquad (i, j = 1, 2)\\
&Q_1^+(Q_1^-Q_1^+ + Q_2^-Q_2^+) = Q_1^+H \\
&Q_2^+(Q_1^-Q_1^+ + Q_2^-Q_2^+) = Q_2^+H.
\end{aligned}
\tag{12}
$$

(plus Hermitian conjugated relations). Other triple products of the generalized supercharges (10) vanish.

Finally, for Λ-type we obtain the supercharges

$$Q_1^+ = \left(\frac{1}{2}\right)^{1/2} \begin{vmatrix} 0 & 0 & 0 \\ p_x - \mathrm{i}\,W_1(x) & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad Q_2^+ = \left(\frac{1}{2}\right)^{1/2} \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & p_y - \mathrm{i}\,W_2(y) \\ 0 & 0 & 0 \end{vmatrix} \qquad (13)$$

and the Hamiltonian

$$H = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}(W_1^2(x) + W_2^2(y)) + \frac{1}{2} \begin{vmatrix} W_2'(x) - W_1'(y) & 0 & 0 \\ 0 & W_1'(x) + W_2'(y) & 0 \\ 0 & 0 & W_1'(x) - W_2'(y) \end{vmatrix} \qquad (14)$$

where $W_1'(x) = \mathrm{d}\,W_1(x)/\mathrm{d}x$ and $W_2'(y) = \mathrm{d}\,W_2(y)/\mathrm{d}y$. The generalized supercharges (13) and the Hamiltonian (14) generate the algebra:

$$\begin{aligned} &Q_i^{-2} = Q_i^{+2} = [H, Q_i^-] = [H, Q_i^+] = 0 \qquad (i, j = 1, 2) \\ &(Q_1^+ Q_1^- + Q_2^+ Q_2^-) Q_1^+ = Q_1^+ H \\ &(Q_1^+ Q_1^- + Q_2^+ Q_2^-) Q_2^+ = Q_2^+ H. \end{aligned} \qquad (15)$$

(plus Hermitian conjugated relations). Other triple products of the generalized supercharges (13) vanish.

Our consideration of an example of the two-dimensional (two bosonic modes) generalized superalgebras, based on three-level systems, shows that in the framework of our approach one can obtain multidimensional generalized superalgebras. The multidimensional generalized superalgebras obtained have a structure differing from one-dimensional analogues [1]. In contrast, in the case of multidimensional extensions within a framework of standard and spin–orbit coupling procedures the structure of the multidimensional superalgebras remains the same as for the one-dimensional superalgebra. The construction of various multidimensional generalized superalgebras, based on the non-interacting N-level system with various numbers of bosonic modes is also possible.

References

[1] Semenov V V and Chumakov S M 1991 *Phys. Lett.* **262B** 451
[2] Witten E 1981 *Nucl. Phys.* B **188** 513
[3] Rubakov V A and Spiridonov V P 1988 *Mod. Phys. Lett.* A **3** 1337
[4] Durand S and Vinet L 1990 *Phys. Lett.* **146A** 299
[5] de Crombrugghe M and Rittenberg V 1983 *Ann. Phys., NY* **151** 99
[6] Balantetekin A B 1985 *Ann. Phys., NY* **164** 277
[7] Gamboa J and Zanelli J 1955 *Phys. Lett.* **165B** 91
[8] Beckers J, Dehin D and Hussin V 1987 *J. Phys. A: Math. Gen.* **20** 1137; 1988 *J. Phys. A: Math. Gen.* **21** 651
[9] Yoo H-I and Eberly J H 1985 *Phys. Rep.* **118** 241